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Regulating Eternal Inflation II
The Great Divide

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Abstract: In a previous paper, two of the authors presented a "regulated" picture of eternal inflation. This picture both suggested and drew support from a conjectured discontinuity in the amplitude for tunneling from positive to negative vacuum energy, as the positive vacuum energy was sent to zero; analytic and numerical arguments supporting this conjecture were given. Here we show that this conjecture is false, but in an interesting way. There are no cases where tunneling amplitudes are discontinuous at vanishing cosmological constant; rather, the space of potentials separates into two regions. In one region decay is strongly suppressed, and the proposed picture of eternal inflation remains viable; sending the (false) vacuum energy to zero in this region results in an absolutely stable asymptotically flat space. In the other region, we argue that the space-time at vanishing cosmological constant is unstable, but *not* asymptotically Minkowski. The consequences of our results for theories of supersymmetry breaking are unchanged.

Keywords: dS vacua in string theory, Solitons Monopoles and Instantons.

Contents

1. Introduction

The possibility that the universe inflates eternally, to create an infinite and complex mixture of causally disconnected inflating and non-inflating regions, is one of the most interesting and perplexing ideas to emerge in cosmology. In a recent paper [1], two of us (TB and MJ) presented a picture of a large class of eternal inflation models that greatly simplifies their analysis by viewing the eternally inflating universe as a finite system comprised of the causal diamond of a single observer.

This picture, which has consequences for the Landscape idea as well as for models of low-energy supersymmetry breaking, both suggested and gained support from an interesting new result in the dynamics of true-vacuum bubble nucleation as described by Euclidean instanton techniques. In particular, it was found that in a certain class of potentials, the instanton action for a transition from positive (false) to negative (true) vacuum energy did *not* tend to infinity as the false vacuum energy V_F was reduced to zero, as would be required to give a finite nucleation probability[1] and hence accord with intuition regarding the decay of Minkowski space to a negative vacuum ("big crunch") space. This result was supported by general analytic arguments, as well as numerical results for $\epsilon \sim 1$, where ϵ controls the scale in field value over which the potential varies. On the basis of these results it was conjectured that

[1] As $V_F \to 0$, the required background subtraction becomes infinite, requiring an infinite instanton action to cancel it and leave a finite decay probability.

1. The same behavior holds at $\epsilon \ll 1$, and
2. for $V_F \equiv 0$, a second (non-compact) instanton, like the one found in the absence of gravity, exists which allows much faster decay, so that
3. for all ϵ there is a discontinuity in the decay rate as $V_F \to 0$.

In this paper, we will demonstrate that while the specific calculations presented in [1] are correct, the above conjecture is not[2]. Instead we find that the space of potentials is partitioned by a *Great Divide*, into one class where Minkowski space is unstable, and a second class where the tunneling rate is indeed suppressed — as argued in [1] — by the factor $e^{-\pi(RM_P)^2}$ (where R is the de Sitter radius corresponding to the false vacuum), and hence vanishes at $V_F = 0$. The stability, for some potentials, of a seemingly metastable Minkowski vacuum was noted long ago by Coleman and De Luccia [3] in the thin-wall limit and subsequently discussed by several authors [4, 5] outside of that limit.

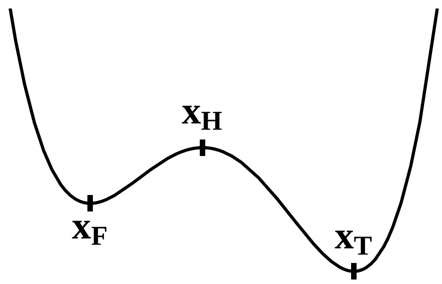

Figure 1: The potential $V(\phi)$, with the true vacuum x_T, the false vacuum x_F and the "Hawking-Moss" point x_H labeled.

In sections 2-4 we will review the instanton formalism, give approximate analytic solutions, then examine the behavior of the instanton solutions in the limit where $V_F \to 0$, using both analytic and numerical techniques. After elucidating the actual behavior of the instantons, we will argue in section 5 that the Great Divide consists precisely of those potentials which, in the $V_F \to 0$ limit, have static domain walls interpolating between the true and false stationary points of the potential[3]; we also argue that the Great Divide is appropriately named because its codimension in the space of potentials is one. In sections 6 and 7, we will discuss our results in connection with the picture of eternal inflation put forward in [1]. In section 6, we will argue that it is inappropriate to think of potentials describing unstable Minkowski space as having to do with quantum gravity in asymptotically flat space, then discuss what they may, instead, correspond to. In section 7, we comment on the implications of potentials above the Great Divide for the string theory landscape. A brief summary of our conclusions is given in section 8.

2. Field equations

In this paper, we will study a single scalar field with potential of the form

$$V(\phi) = \mu^4 v\,(\phi/M)\,, \tag{2.1}$$

[2] R. Bousso, B. Freivogel and M. Lippert, have discovered this fact independently [2].

[3] This observation is related to the work of Cvetic *et. al.* on singular domain walls and their relation to CDL bubbles [6].

where, defining $x \equiv \phi/M$, the dimensionless potential $v(x)$ is given by

$$v\left(x\right)=f\left(x\right)-\left(1+z\right)f\left(x_{F}\right), \tag{2.2}$$

where here and henceforth subscripts "T" and "F" will label values at the true and false vacuum, respectively (see figure 1), and where

$$f(x)=\frac{1}{4}x^{4}-\frac{b}{3}x^{3}-\frac{1}{2}x^{2}. \tag{2.3}$$

We will tune the parameter b such that the potential has three extrema as shown in figure 1, and has variations of order 1 between x_F and x_T. The non-negative parameter z controls the false vacuum cosmological constant V_F, so that $V_F \to 0$ as $z \longrightarrow 0$.[4] The general scaling form of the potential is motivated by considerations of naturalness. Typical potentials which cannot be fit into this form have fine-tuned dimensionless coefficients and are not stable to radiative corrections.[5]

For many choices of the parameters b and z, there will be 0(4) invariant instantons, which travel between the basins of attraction of the minima at x_T and x_F. Together with a scalar field configuration, $\phi(z)$, the instanton is described by an Euclidean manifold of the form

$$ds^{2}=dz^{2}+\rho^{2}(z)d\Omega^{2}, \tag{2.4}$$

where $d\Omega^2$ is the surface element of a unit 3-sphere. Defining the following dimensionless variables:

$$r\equiv\frac{\mu^{2}\rho}{M}, \tag{2.5}$$

$$s\equiv\frac{\mu^{2}z}{M}, \tag{2.6}$$

$$\epsilon^{2}\equiv\frac{8\pi M^{2}}{3M_{P}^{2}}, \tag{2.7}$$

the coupled Euclidean scalar field and Einstein's equations are

$$\ddot{x}+\frac{3\dot{r}}{r}\dot{x}+u'=0, \tag{2.8}$$

$$\dot{r}^{2}=1+\epsilon^{2}r^{2}E, \tag{2.9}$$

where $u(x) \equiv -v(x)$, primes and dots, respectively, refer to $x-$ and $s-$derivatives, and E is the Euclidean energy of the field, defined as

$$E=\frac{1}{2}\dot{x}^{2}+u(x). \tag{2.10}$$

[4] The way in which we have chosen to tune the vacuum energy is not really appropriate in many supergravity models. There, one tunes a constant in the superpotential. If there are excursions in field space of order m_P, this changes the potential in a more complicated way than a simple subtraction. We hope to return to a study of supergravity models in a future publication.

[5] The major exception we know of is the case of moduli in string theory near singular points in moduli space: while the typical potential for moduli depends on ϕ/m_P or ϕ/m_S, near singular points (where other degrees of freedom become light) the potential can have more rapid variation.

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For future reference, the dynamics of the Euclidean energy are determined by the equation

$$\dot{E} = -3\frac{\dot{r}}{r}\dot{x}^2. \tag{2.11}$$

When the false vacuum well has positive energy, the Euclidean spacetime of eq. 2.4 is necessarily compact, spanning an interval between $s = 0$ and $s = s_{\max}$. To avoid singular solutions to eq. 2.8, the field must have zero derivative (i.e. $\dot{x} = 0$) at $s = 0$ and $s = s_{\max}$. There will thus be a non-singular solution to the instanton equations if the boundary conditions

$$r(0) = 0, \quad r(s_{\max}) = 0, \quad \dot{x}(0) = 0, \quad \dot{x}(s_{\max}) = 0, \tag{2.12}$$

can be met for some set of endpoints in the evolution of x near x_T and x_F. Solutions with two zeros in $\dot{x}$ will be referred to as "single-pass" instantons. We also note [5] that multifield models can be studied using these methods as well, as long as we restrict attention to instantons for which $\dot{\phi}^i = 0$ only at two points. In that case, however, one might be interested in potentials with more minima and maxima.

The decay rate of the false vacuum is given by

$$\Gamma = Ae^{-S_E}, \tag{2.13}$$

where A is a pre-factor that will be neglected in what follows. The total Euclidean action, S_E, is the difference between the action of the instanton, S_I, (which is negative due to the positive curvature of the instanton) and the action of the background spacetime, $S_{\rm BG}$ (which is negative and larger in magnitude than the instanton action)

$$S_E = S_I - S_{\rm BG}. \tag{2.14}$$

The instanton action is given by

$$S_I = -4\pi^2\left(\frac{M^4}{\mu^4}\right)\int_{s=0}^{s=s_{\max}} ds\left(r^3 u + \frac{r}{\epsilon^2}\right). \tag{2.15}$$

The background subtraction term (for an end-point of the evolution in x near x_F) is given by

$$S_{\rm BG} = \frac{8\pi^2}{3\epsilon^4 u_F}. \tag{2.16}$$

In what follows we will be interested in the relative magnitude of the instanton and background actions. In particular, when the false vacuum cosmological constant is taken to zero, the backgound subtraction term eq. 2.16 diverges. Unless the instanton action scales similarly, the tunneling rate is very strongly suppressed for small u_F.

3. Approximate analytic solutions

We can solve eq. 2.8 and 2.9 exactly when the Euclidean energy remains approximately constant for a period of time. This can only occur in the neighborhood of the extrema of the potential. The focus of this study is on transitions from a positive Euclidean energy well at x_T to a negative Euclidean energy well at x_F, but the results we present below can be used to study arbitrary combinations of positive and negative energy wells. The approximate solution to the instanton equations near x_H (see figure 1) was presented in [7], and is relevant for the study of oscillating solutions.

Consider the evolution of the field in the neighborhood of x_T or x_F. The field will begin/end with zero velocity and some displacement, $\delta_{T,F}$, from x_T or x_F. If the variable $\delta_{T,F}$ is small, then the field will loiter in the neighborhood of the maximum. During this time, the Euclidean energy of the field will remain roughly constant and, if the velocity remains small, equal to the value of u at the maximum. Equation 2.9, for the cases of loitering near the true or false vacuum maxima, then reduces to

$$\dot{r}^2 \simeq 1 + \epsilon^2 r^2 u_{T,F}, \tag{3.1}$$

which can be integrated to yield

$$r(s) = \frac{1}{\epsilon\sqrt{-u_{T,F}}} \sin\left(\epsilon\sqrt{-u_{T,F}}\right). \tag{3.2}$$

If we take the false vacuum maximum to have $u_F < 0$, then we can recognize this as the metric for Euclidean de Sitter space (the four sphere). Substituting Eq 3.2 into eq. 2.8 yields:

$$\ddot{x} + 3\epsilon\sqrt{-u_{T,F}} \cot\left(\epsilon\sqrt{-u_{T,F}} s\right) \dot{x} + u'(x) = 0. \tag{3.3}$$

Since we are trying to find solutions only in the vicinity of the true and false vacuum maxima, we may Taylor expand the potential about $x_{T,F}$, keeping only the constant and quadratic terms. After making the change of variables $y = \cos\left(\epsilon\sqrt{-u_{T,F}}\right)$ and $\delta = x - x_{T,F}$, we then obtain

$$\left(1 - y^2\right) \frac{d^2\delta}{dy^2} - 4y\frac{d\delta}{dy} + \frac{\omega^2}{\epsilon^2 u_{T,F}}\delta = 0, \tag{3.4}$$

where $\omega^2 \equiv |u''_{T,F}|$. This can be recognized as the hyperspherical differential equation, the solution of which is given in terms of Legendre functions. After imposing the boundary conditions $\dot{\delta}(y = 1) = 0$ and $\delta(y = 1) = \delta_{T,F}$, we obtain

$$\delta(y) = \frac{-2i\delta_{T,F}}{\nu\left(\nu + 1\right)} \left(y^2 - 1\right)^{-1/2} P^1_\nu(y), \tag{3.5}$$

with

$$\nu = -\frac{1}{2}\left(1 + \sqrt{9 + \frac{4\omega^2}{\epsilon^2 u_{T,F}}}\right). \tag{3.6}$$

For $s \ll \epsilon\sqrt{|u_{T,F}|}$, this solution can be written in terms of Bessel functions.

We have found an approximate analytic solution near the true and false vacuum maxima. However, in order to construct the entire single-pass instanton we must evolve across regions of the potential in which our approximations break down. This requires a numerical approach, which will be presented in section 4.2.

4. The $V_F \to 0$ limit

We are now in a position to re-examine some of the conclusions of [1]. Two of the authors (TB and MJ) conjectured that for all ϵ the instanton describing a transition from a positive energy false vacuum to a negative energy true vacuum approaches a finite size as $z \to 0$, and therefore the instanton action would not scale with the background subtraction term. We argued (to ourselves) that there would also be a flat space instanton which existed for $z = 0$, by a version of Coleman's overshoot/undershoot argument. This implied a discontinuous limit as the false vacuum energy was sent to zero.

Here, we will present numerical and analytical arguments that below some (potential dependent) ϵ_c there are in fact large dS instantons that asymptote as $z \to 0$ to the flat space instanton. Above ϵ_c, there are finite-size instantons with finite action as $z \to 0$, but no flat space instanton. At ϵ_c (on the Great Divide), we will find that the instanton for $z = 0$ is a static domain wall solution of the coupled Euclidean Einstein and field equations.

4.1 Small ϵ

Let us explore the small ϵ case first, and argue that if a single-pass instanton exists, it *must* resemble the dimensionless de Sitter metric, eq. 3.2, over most of its volume. From eq. 2.9, we see that the Euclidean energy, which is bounded from below by the value $u(x_H)$ of the potential at the Hawking-Moss maximum, must be negative for a turn-around in r to occur. If there is a turn-around, the value of r at this point, r_m, will be

$$r_m = \frac{1}{\epsilon\sqrt{-E_m}}. \tag{4.1}$$

Since the Euclidean energy is bounded, as ϵ is decreased, r_m must increase. If there is a compact nonsingular instanton, the field must evolve in such a way to facilitate this growth in r. When the field is not in the vicinity of the extrema of the potential, it will move between the potential wells in a time of order one. During this time, r will grow to some ϵ independent size. Thus, for r to become large enough to find a turn-around in the small ϵ limit, the field must loiter in the vicinity of one of the extrema of the potential.

Loitering near the Hawking-Moss maximum leads to an oscillatory motion, because this is a minimum of the Euclidean potential. There are non-singular solutions which make of order $\frac{1}{\epsilon}$ oscillations before ending up in the basin of x_F. These are not single pass instantons. Loitering near the true vacuum maximum will cause r to grow as in eq. 3.2 (linearly if $s \ll \epsilon\sqrt{u_T}$). However, because the friction term decays during the loitering phase, these solutions will in general have too much energy and overshoot the false vacuum maximum. For intermediate values of ϵ, the growth in r near the true vacuum becomes important, as we will see below.

The only viable option is then that the field be near x_F at the turn-around in r. If we take the end-point near x_F to be at $s = 0$, the field must remain near x_F until $r = r_m$. This evolution should be well described by the analytic solution eq. 3.5 derived in the previous section. The Euclidean energy at r_m will be given by

$$E_m \simeq u_F + \frac{1}{2}\dot{\delta}_m^2 - \frac{\omega^2}{2}\delta_m^2. \tag{4.2}$$

We can write δ_m and $\dot{\delta}_m$ in terms of Gamma functions

$$\delta_m = \delta(s = \pi/2\epsilon\sqrt{v_F}) = \delta_F \frac{\sqrt{\pi}}{\Gamma\left(1 - \frac{\nu}{2}\right)\Gamma\left(\frac{3}{2} + \frac{\nu}{2}\right)}, \tag{4.3}$$

and

$$\dot{\delta}_m = \dot{\delta}(s = \pi/2\epsilon\sqrt{v_F}) = -\delta_F \epsilon\sqrt{\pi v_F} \frac{2+\nu}{\Gamma\left(\frac{1}{2} - \frac{\nu}{2}\right)\Gamma\left(2 + \frac{\nu}{2}\right)}. \tag{4.4}$$

This limit will be an important component of the numerical scheme presented in the following section. We note that δ_m and $\dot{\delta}_m$ are of the same order of magnitude, and must be much smaller in magnitude than v_F for our approximation scheme to remain self-consistent. This can always be arranged by making δ_F of order $\exp(\frac{-1}{\epsilon\sqrt{v_F}})$. Thus, we can see that there is a self-consistent solution in the vicinity of x_F which tracks the de Sitter solution until r_m.

In fact, it is necessary, for small ϵ, to choose δ_F small enough that the de Sitter/Legendre approximation remains valid until $s = \frac{\pi}{\epsilon\sqrt{v_F}} - o(1)$. If we do not do this, then $x(s)$ moves rapidly away from x_F on a time scale of $o(1)$, while $r(s)$ is still $\gg 1$. It will either overshoot x_T or stop and fall back, long before the second zero of $r(s)$ is reached. In neither case do we get a single pass instanton. The rest of the instanton consists of a traverse from the vicinity of the false vacuum, to the basin of attraction of the true vacuum, in a time of $o(1)$ (ϵ-independent for small ϵ). It is important that, since $r \ll 1/\epsilon$ during this traverse, eq. 3.1 indicates that $r(s)$ is approximately linear in this period, and indeed also linear for a long period before $x(s)$ leaves the vicinity of the false vacuum.

It is convenient to think of the rest of the instanton as a function of a new time variable t which starts at $t = 0$ near the true vacuum and increases toward the false vacuum so that $d/dt \equiv -d/ds$. Since $r(t) \approx t$ when $r \ll 1/\epsilon$, we have

$$\frac{d^2x}{dt^2} + \frac{3}{t}\frac{dx}{dt} = -u'(x), \tag{4.5}$$

with the boundary conditions $\frac{dx}{dt}(t = 0) = 0$ and $x_H < x(t = 0) < x_T$.

This equation is just the equation for an instanton in quantum field theory, neglecting gravitational effects. Coleman [8] showed that one can find solutions which start in the basin of attraction of the true minimum, and get arbitrarily close to (or even overshoot) the false minimum. Eq. 4.5 is ϵ-independent, but as ϵ goes to zero, the range of t over which it is a good approximation to the real instanton solution grows as $1/\epsilon$. Thus, for small enough ϵ, we can use Coleman's argument to show that there are solutions of eq. 4.5, which are non-singular at $t = 0$ and penetrate into the region where the Legendre approximation is valid. By varying the initial position $x(t = 0)$ among all such solutions, we can tune the logarithmic derivative of x at a given point t^* where both approximations are valid, within a finite range.

The conditions that the two solutions match at some point (t^*, s^*) are

$$t^* = \frac{1}{\epsilon\sqrt{v_F}} \sin(\epsilon\sqrt{v_F} s^*), \tag{4.6}$$

$$\frac{1}{x(s^*)}\frac{dx}{ds} = -\frac{1}{x(t^*)}\frac{dx}{dt}, \tag{4.7}$$

$$x(s^*) = x(t^*), \tag{4.8}$$

where functions of s^* are in the de Sitter/Legendre approximation and functions of t^* are in the zero-gravity approximation. Once we know that there is a range of $x(t = 0)$ for which $x(t)$ penetrates into the range where the Legendre approximation is valid, we can tune $x(s = 0)$ to satisfy the last condition. We know that $s*$ is large for very small ϵ, of order $\frac{\pi}{\epsilon\sqrt{v_F}} - o(1)$, in which case the first condition becomes $t* = s*$.

$x(t = 0)$ is then tuned to match the logarithmic derivatives. Although there is a range of s over which $x(s)$ is rapidly varying, its logarithmic derivative is roughly constant over that range. The only place where the logarithmic derivative is large, is near the second zero of the sine, but for small ϵ the matching occurs far from that region (t^* large but $\ll \frac{1}{\epsilon\sqrt{v_F}}$). It is thus plausible that by varying s^* and $x(t = 0)$ we can satisfy both of Equations 4.6 and 4.7. If this is the case, then a non-singular, large radius instanton exists. As $v_F \to 0$, this goes over smoothly to an "instanton for the decay of asymptotically flat space".

The argument above indicates the possibility of a true asymptotic matching of solutions of the non-gravitational equations to solutions of the de Sitter/Legendre approximation over a range of s which grows as $\epsilon \to 0$. Since we cannot exhibit solutions of the non-gravitational equations exactly, our argument is not completely rigorous. In the next section we will present numerical calculations, which show that it is correct.

4.2 Numerical results for small ϵ

To confirm the validity of the conclusions above, we have undertaken a semi-analytic search for single pass instantons in a potential with a positive false vacuum and a negative true vacuum. Here, we will focus on the potential shown in figure 2, though qualitatively our results are potential independent (we have confirmed this by studying a variety of potentials).

The strategy is to use the matching scheme discussed in section 4.1. We will relax the zero-gravity approximation for the evolution from the true vacuum well to the false vacuum well, and numerically evolve eqs. 2.9 and 2.8. To fix the initial conditions of the numerical evolution from the true vacuum side of the potential, we will use an analytic solution to evolve for the first time step. If it is near x_T, we use eq. 3.5; if not, we approximate the potential as linear, yielding a $\delta(s) \propto s^2$. We then evolve and attempt to match onto the de Sitter/Legendre approximation (eq. 3.2 and 3.5) when the field approaches x_F. Of course, we are not guaranteed to find a match for all ϵ. It was shown by Coleman and De Luccia [3], that in the thin-wall limit there are cases where the transition from a positive (Euclidean) energy well to a zero energy well is forbidden. This occurs when the positive energy at the true vacuum maximum becomes too small, so that an over-shoot solution becomes impossible. This

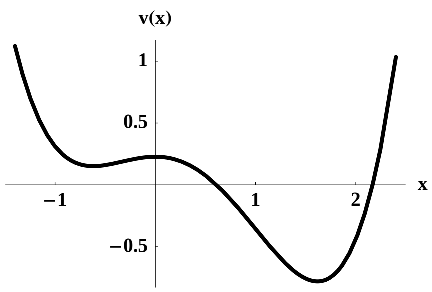

Figure 2: The potential, $v(x)$, used for the numerics. The parameter b is fixed at $b = 1$, and z will be allowed to vary (this plot shows $z = 1$).

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would prevent the instanton from ever entering a regime where the de Sitter/Legendre approximation was valid.

The need for a semi-analytic approach is evident from the fantastically small displacement from the false vacuum required to find solutions with large r_m. Numerically evolving the solution over the entire trajectory would become impossible as the field approaches x_F. Also for reasons of numerical tractability, we match the solutions at at r_m, where $s = \pi/(2\epsilon\sqrt{v_F})$, and the Legendre function can be written in terms of (calculable) $\Gamma-$functions as in eq. 4.3 and 4.4.

This method also has its limitations. For small enough $\epsilon\sqrt{v_F}$, we may be trying to compare field velocities at a precision that is not achievable by the numerical integrator. Despite these difficulties, we have been able to construct a number of instantons in the intermediate ϵ regime, examples of which are shown in figure 3. It can be seen in this plot that as $z \to 0$, these instantons are growing. Since we have shown that a matching is possible at r_m, as $v_F \to 0$, by the argument given in section 4.1, these instantons must scale with the background subtraction term.

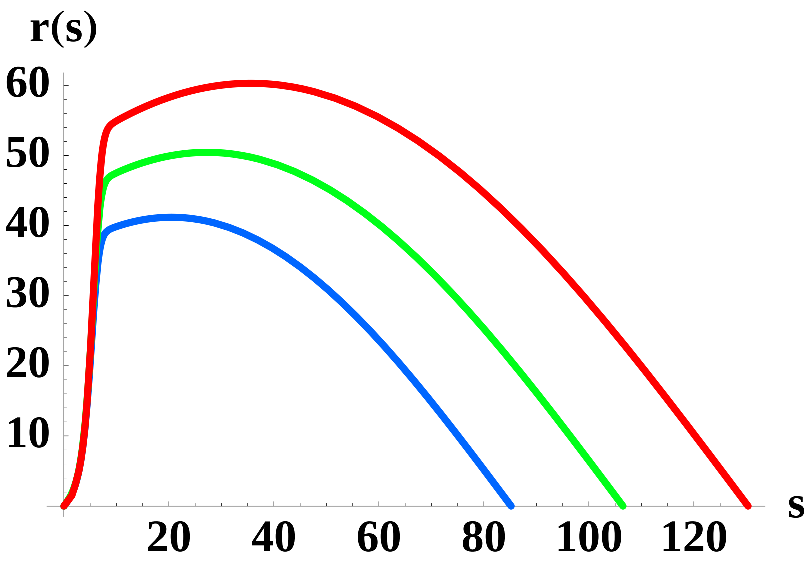

Figure 3: Evolution of $r(s)$ for $\epsilon = .72$ and $z = (.01, .008, .006)$ from bottom to top. The matching between the analytic and numeric solutions occurs at the maximum of r, r_m.

4.3 Large ϵ

To study large[6] values of ϵ, where the approximations introduced above are not necessarily valid, we must take an entirely numerical approach. We choose to begin the evolution from the true vacuum side of the potential, varying δ_T until a solution is found. To fix the initial conditions of the numerics, we will again use an analytic solution to evolve for the first time step as described in the previous section.

[6] By large we mean of order one. While the formalism will accommodate arbitrarily large values of ϵ, there will be an ϵ after which only the Hawking-Moss instanton exists.

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Shown in figure 4 is the evolution in x for $\epsilon = .85$ as $z \to 0$. Shown in figure 5 is the evolution in r with the same parameters. It can be seen that as $z \to 0$, the instanton approaches a constant, finite size. Therefore, for large ϵ, the instanton action will not scale with the background subtraction term.

To discuss the continuity of the limit $V_F \to 0$, we must first determine in which cases there is an instanton for $V_F = 0$. If this instanton describes the decay of a spacetime with exactly zero cosmological constant, then the evolution in r must be from $r(s = 0) = 0$ to $r(s = \infty) = \infty$. The field will be moving from some initial position near x_T at $s = 0$ to *exactly* x_F at $s = \infty$. If, starting near x_T, there is a region of δ_T-space in which over-shoot occurs, then there must be a second zero in $\dot{x}$. The question is then what value r takes at the second zero of $\dot{x}$.

In all of the numerical examples we have studied with $z = 0$, we find that $r = 0$ at the second zero of $\dot{x}$. The turn-around in r in these cases is not caused by loitering in the vicinity of a negative energy extremum of the potential. Instead, as the field is climbing towards x_F, the negative potential energy comes to dominate over the kinetic energy. Since ϵ is rather large, r does not need to grow very large to cause a turn-around in r. Since the end-points of this instanton are on the boundaries of the unique over- and under-shoot regions of the potential, there is no other single-pass instanton with $r(s = \infty) = \infty$.

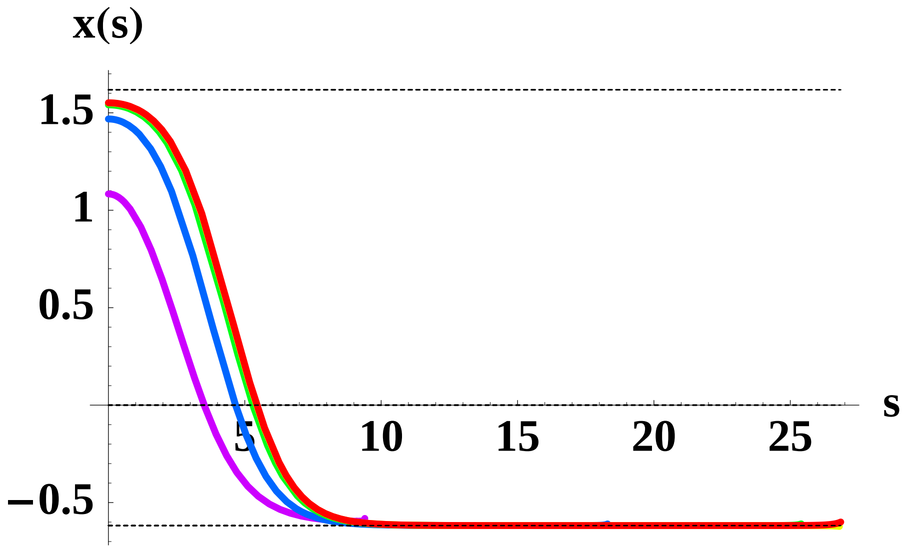

Figure 4: The evolution of $x(s)$ for $\epsilon = .85$ and $z = (1, .1, .01, .001, .0001)$ from bottom to top. The dashed horizontal lines indicate the positions x_T (top) and x_F (bottom) .

5. The Great Divide

In this section we show that, for any potential $v(x)$, there is a critical value of ϵ for which planar domain wall solutions exist. As one goes from the small to the large ϵ regime, there is a transition point between the two behaviors discussed in section 4. We will define ϵ_c as the transition point in the case where $z = 0$ (when the false vacuum well has zero energy).

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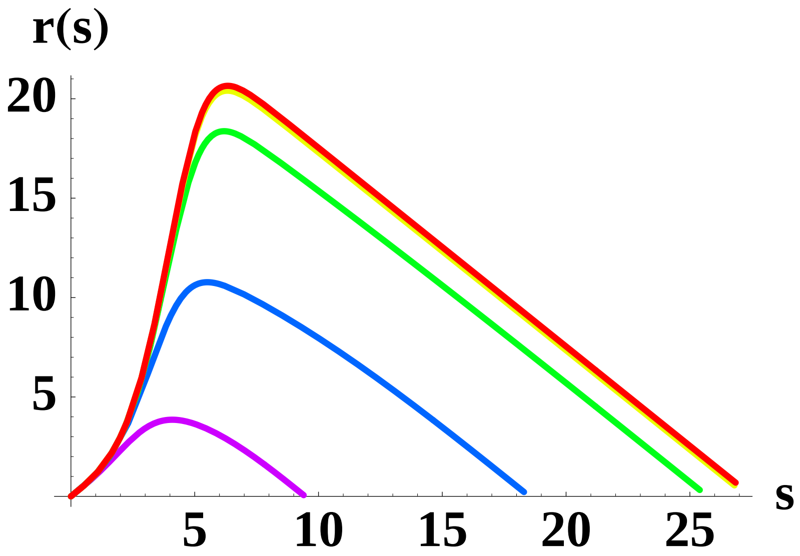

Figure 5: The evolution of $r(s)$ for $\epsilon = .85$ and $z = (1, .1, .01, .001, .0001)$ from bottom to top.

We have found instantons (with $z = 0$) for a variety of ϵ near ϵ_c as shown in figure 6. The evolution of the field is from the vicinity of x_T at $s = 0$ to x_F at $s = \infty$. Of course, we cannot track the entire evolution, but we can follow it for some finite time scale by tuning δ_T to approach the boundary between the under- and over-shoot solutions. It can be seen from these numerical examples that r is growing very large in the vicinity of the true vacuum.

As we approach ϵ_c, the initial displacement on the true vacuum side, δ_T, is decreasing as shown in figure 7. Because we are starting with more energy on the true vacuum side of the potential, we must send $\delta_F \to 0$ as well. Therefore, at this critical value of ϵ, the instanton interpolates exactly between x_T at $s = -\infty$ and x_F at $s = +\infty$. Also, note that after we analytically continue to the Lorentzian solution, the interior of the CDL bubble will be infinitely large. This solution therefore describes a static domain wall.

We can understand this behavior by looking at the energetics of the evolution from x_T to x_F. The instanton equations in the critical limit approach the static domain wall equations

$$\ddot{x} + \frac{3\dot{r}}{r}\dot{x} + u' = 0, \tag{5.1}$$

$$\dot{r}^2 = \epsilon^2 r^2 E, \tag{5.2}$$

s now runs between $-\infty$ and ∞, and a domain wall solution asymptotes to the two vacua on opposite sides. The energy is always decreasing along the trajectory from the true to the false vacuum well. The question is whether x can lose just enough energy during its traverse to asymptote to x_F without overshooting. If $\epsilon = 0$ the answer is clearly no, because energy is conserved. The solution overshoots the false vacuum. This persists for very small ϵ. On the other hand, in the mathematical limit $\epsilon \gg 1$, the friction term dominates the motion and x undershoots in a finite time. It follows that there is a critical value of ϵ where

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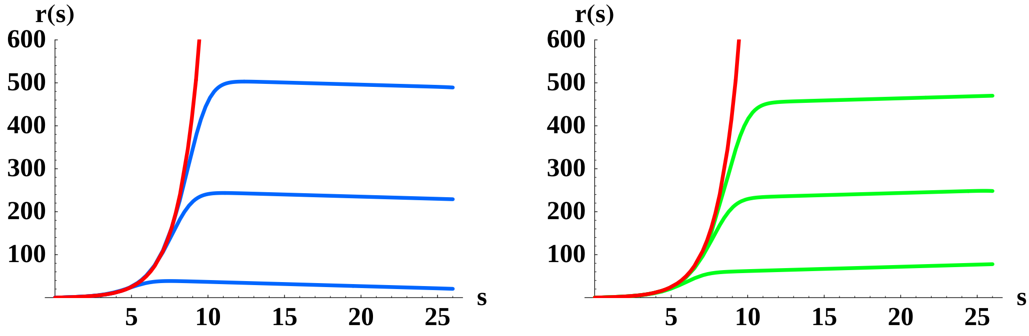

Figure 6: The evolution of $r(s)$ for $z = 0$ on either side of ϵ_c. Shown on the left are values of $\epsilon > \epsilon_c$ in blue ($\epsilon = (.8, .75, .745)$ from bottom to top) and $\epsilon_C \sim .74$ in red. The instantons with $\epsilon > \epsilon_c$ are compact, having two zeros in r. On the right are values of $\epsilon < \epsilon_c$ ($\epsilon = (.7, .73, .735)$ from bottom to top) in green and and ϵ_c in red. The instantons with $\epsilon < \epsilon_c$ are not compact, with $r \to \infty$ as $s \to \infty$.

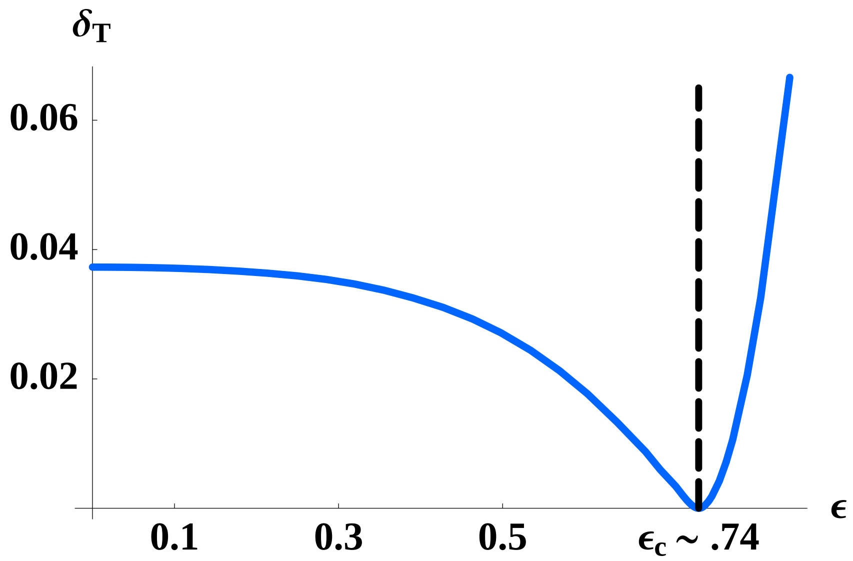

Figure 7: It can be seen in this plot of δ_T vs ϵ for the case where $z = 0$ that there is an ϵ_c for which $\delta_T \to 0$. Below this value, δ_T is approaching the zero-gravity solution, and above it, $\delta_T \to x_T - x_H$.

x indeed asymptotes to x_T and we have a static domain wall solution in the presence of gravity. The critical value is clearly $o(1)$. Since we have found such a solution by tuning a single parameter, the codimension of the subset of potentials which have a domain wall is 1, and the subset forms a Great Divide in the space of potentials.

We have shown both that there is a critical value of ϵ at which domain walls exists, and that the flat space instanton solution, which exists below the Divide, approaches the domain wall solution at this critical value. Above the divide, the flat space instanton and the associated large instantons for small v_F, disappear. Flat space is stable, and the stability of nearly flat dS spaces has a clear entropic explanation.

6. Below the great divide

In [1], along with the conjecture of a discontinuity of the tunneling action at $V_F \to 0$ came a (retrospectively flawed) physical argument to explain the discontinuity, based on the physical picture of quantized dS space adumbrated in [9]. In that picture, quantized dS space is equipped with two operators: the static Hamiltonian H, and the Poincare Hamiltonian P_0; these satisfy a finite-dimensional approximation to the commutation relation

$$[H, P_0] \sim \frac{1}{R} P_0, \tag{6.1}$$

where R is the de Sitter radius. The eigenvalues of H are highly degenerate, and bounded by something of order the dS temperature, $T_{dS} = \frac{1}{2\pi R}$. The low-lying eigenstates of P_0 are metastable (when evolved using H), and correspond to states localized in a given horizon volume; the lowest lying eigenstates have small degeneracies, and the ground state is unique. The conjectured discontinuity in the tunneling probability was alleged to be related to the fact that the for finite V_F the CDL instanton describes the decay of the thermal ensemble of H eigenstates (a system of high-entropy), but that for vanishing V_F it describes the decay of a low-entropy system consisting just of the single P_0 ground state.

The flaw in this argument is that it hypothesizes both a stable P_0 eigenstate, and also the decay of that stable system. That is, the existence of the CDL instanton for potentials below the great divide is, in fact, evidence that these low energy effective theories do not correspond to limits of theories describing asymptotically flat space-time.

The conformal boundary of the Lorentzian continuation of the CDL instanton is not the same as that of Minkowski space: in the usual parametrization (u, Ω) of future null infinity, $\mathcal{I}^+$, in terms of a null coordinate u and a transverse sphere, the boundary becomes geodesically incomplete because the asymptotic bubble wall hits $\mathcal{I}^+$ at a finite value of u. Neither the Lorentz group (consisting of the conformal group of the sphere accompanied by a rescaling of u) nor the time translation group (the generator of which is just $P_0 = \frac{\partial}{\partial u}$, in a particular Lorentz frame) is an asymptotic symmetry of this spacetime. Thus, the "explanation" of an hypothetical discontinuity in [1] was based on an equally hypothetical operator P_0. Neither exists.

If potentials below the Great Divide do not correspond to effective theories of gravity in asymptotically flat space, what do they correspond to? Two possibilities consistent with the authors' current understanding of quantum gravity are:

1. Nothing. That is, there simply are no theories of quantum gravity which give rise to such potentials.

2. These theories correspond to models of quantum gravity which, in the $V_F \to 0$ limit under consideration, actually contain only a finite number of excitations of the Minkowski solution. This would remove the apparent contradiction between the infinite number of states of the would-be asymptotically flat space and the finitely bounded entropy of the maximal-area causal diamond in the Big Crunch.

The confusion may be simplified enormously if the conjecture of [10] is accepted. According to that hypotheses, the only viable quantum theories of asymptotically flat space time are exactly supersymmetric, and all models with a vacuum energy that can be tuned to be arbitrarily small become exactly supersymmetric in that limit. At the moment, this conjecture is valid for all models which have been derived from string theory in a reliable manner. The whole concept of the Great Divide is defined in terms of one-parameter families of potentials, with vacuum energy that can be tuned to zero. The conjecture of [10] thus implies that all valid models of quantum gravity will fall above the Great Divide; which is hypothesis 1 above.

7. Connections with eternal inflation

In [1], two of the authors proposed a regulated model of eternal inflation for potential landscapes with only non-vanishing vacuum energies. According to that model the system has a finite number of quantum states, and for most of its time evolution it resembles the dS space of lowest positive vacuum energy[7]. This model remains valid for potentials above the Great Divide. For such potentials, tunneling amplitudes out of dS space are suppressed in a way which may be attributable to the principle of detailed balance, and entropic effects.

To be more precise, for a potential with multiple minima, if the minimum with smallest positive vacuum energy becomes absolutely stable as that vacuum energy is tuned to zero by subtraction, then the CDL tunneling amplitudes are consistent with an interpretation in which eternal inflation on the potential landscape is a finite-dimensional quantum system, most of states of which resemble the dS space of smallest positive vacuum energy. "Decays" of this state into negative vacuum energy Big Crunch regions will occur, but in such an interpretation would be viewed as improbable, low entropy fluctuations of a system that spends most of its time as a large radius dS space.

We also want to comment on the remark of [2] that this kind of landscape is ruled out by observation. This is based on the paper of Dyson *et. al* [11], which is itself a variation on the "Boltzmann's brain" paradox. This paradox arises if we attribute a state in our past to a downward fluctuation from a high-entropy state. It would then be much more probable for our past (given what we observe now) to consist of a *smaller* fluctuation downward in entropy into the universe ten minutes ago. [8]

References [11] and [2] argue that this paradox is not solved by a model in which the universe is a random fluctuation of a finite system with time-independent Hamiltonian. Even if tunneling to negative vacuum energy is suppressed as above the Great Divide, however, there are a number of possible resolutions to this paradox, some well-developed, which we will enumerate here:

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[7]If the minimum with lowest absolute value of the vacuum energy is negative, then this statement might be corrected to "for most of the period during which local observers exist it resembles the dS space of lowest positive vacuum energy".

[8]This paradox is closely related to the observation stressed by Penrose [12] (among others) that there is a contradiction between the claim that the initial conditions for the universe are "generic" (as often claimed is allowed by inflation) and the obseravtion that they are of lower-entropy than the current universe (as demonstrated by the second law of thermodynamics).

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- The transitions rates between vacua may not obey detailed balance [13]. This would be the case if the Farhi-Guth-Guven tunneling mechanism is allowed, and would subvert the paradox by allowing a small inflating region to form, with relatively high probability. This region would then create a large, low-entropy region. Whether the Farhi-Guth-Guven process actually occurs, however, is not clear (see, e.g., [14, 15]).
- The description of the universe as a finite system that can equilibrate is insufficient. Since the region within the horizon *does* apparently approach equilibrium, this would indicate that regions outside the horizon must be taken into consideration in the overall predictions of the theory. In this view, inflation, while difficult to start from a low-energy vacuum, would "get credit" for creating a huge number of observers, so that most observers see inflation in their past. The following argument suggests that there is something wrong with the "causal patch" picture. Consider a multi-vacuum system with a vanishing lowest vacuum energy. According to [2], no paradox arises because tunneling out of the zero-energy vacuum is completely suppressed. Then it would be hard to see how, if the minimal vacuum energy were tuned upward by an infinitesimal amount, this could discontinuously change the observables so that the theory would be ruled out. Continuity as $\Lambda \to 0$ (for which the authors have a greater respect than ever before) implies that either the paradox arises in both cases, or in neither.
- The Hamiltonian of the universe may be time dependent and only asymptote to the static Hamiltonian of the dS observer. A particular model of this is holographic cosmology [16], where, at early times the Hamiltonian does not couple the degrees of freedom within the particle horizon of an observer to those outside it. This is the way in which a non-local, holographic theory can be compatible with the idea of a particle horizon. In such a theory, time has a beginning, and the first recurrence time of an asymptotically dS universe is special since its evolution is not governed by the static dS Hamiltonian. It might be that the explanation of what we see today depends crucially on the time dependence of the dynamics of the early universe. Further recurrences might never produce a universe remotely like our own and might be argued to be irrelevant. The picture of our past as a low entropy fluctuation of a time independent system is what leads to the Boltzmann's brain paradox. It may simply be wrong. This is not a claim that holographic cosmology has (as yet) solved the Boltzmann's brain paradox, but merely that the solution might involve time-dependent dynamics in the early universe.

8. Conclusions

We have seen that there is a rich variety of behaviors of instantons describing the transition from positive or zero energy false vacuum to a negative energy Big Crunch. The complete picture is more detailed than was conjectured in [1], and different than the conventional (thin-wall) wisdom suggests. For small values of ϵ, we have shown that there *does* exist an instanton which resembles Euclidean de Sitter over most of its volume. As the false vacuum energy is taken to zero, the instanton action scales with the background subtraction, and there is no discontinuity in the tunneling rate. However, the analytically continued bubble wall removes a section of the conformal boundary of Minkowski space, providing evidence that low energy effective theories with small ϵ do not correspond to limits of theories describing truly asymptotically flat space-time.

We have found that there exists a static domain wall solution at a critical value of ϵ (ϵ_c). The critical value of ϵ corresponds to a Great Divide in the space of potentials, of codimension one. Below ϵ_c, we find the behavior described in the previous paragraph. Above this value of ϵ, we find compact instantons which *do not* resemble Euclidean de Sitter. The instanton action approaches a constant as the false vacuum energy goes to zero, but the discontinuity claimed in [1] does not exist. We find that there is no non-compact instanton describing the decay of the zero-energy false vacuum, and therefore as the false vacuum energy is decreased, the diverging background subtraction will cause an infinite suppression of the tunneling rate.

The other observation of [1] which remains unchanged by our new results is the remark that metastable SUSY violating vacua of flat space field theories can be viable models of the real world, within the context of Cosmological SUSY Breaking. That is, if we assume that the vacuum energy is tunable and that the limit of vanishing vacuum energy is a supersymmetric theory in asymptotically flat space, then we are above the Great Divide. For finite Λ the probability for the meta-stable vacuum to make a transition to a Big Crunch is of order $e^{-\pi(RM_P)^2}$. This is not a decay, and it has no phenomenological relevance.

Our new results raise interesting questions about the interpretation of models below the Great Divide. The study of these models will be the subject of a future paper.

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References

[1] T. Banks and M. Johnson, *Regulating eternal inflation*, `hep-th/0512141`.

[2] R. Bousso, B. Freivogel and M. Lippert, *Probabilities in the landscape: the decay of nearly flat space*, `hep-th/0603105`.

[3] S. Coleman, F. DeLuccia, *Gravitational effects on and of vacuum decay*, *Phys. Rev.* **D 21** (1980) 3305.

[4] D.A. Samuel and W.A. Hiscock, *Effect of gravity on false vacuum decay rates for O(4) symmetric bubble nucleation*, *Phys. Rev.* **D 44** (1991) 3052.

[5] T. Banks, *Heretics of the false vacuum: gravitational effects on and of vacuum decay, II*, `hep-th/0211160`.

JHEP08(2006)065

[6] M. Cvetič, *Stability of vacua and domain walls in supergravity and superstring theory*, hep-th/9210006;
M. Cvetič, S. Griffies and S.-J. Rey, *Static domain walls in $N = 1$ supergravity*, *Nucl. Phys.* **B 381** (1992) 301 [hep-th/9201007];
M. Cvetič, S. Griffies and H.H. Soleng, *Nonextreme and ultraextreme domain walls and their global space-times*, *Phys. Rev. Lett.* **71** (1993) 670 [hep-th/9212020];
M. Cvetič, S. Griffies and H.H. Soleng, *Local and global gravitational aspects of domain wall space-times*, *Phys. Rev.* **D 48** (1993) 2613 [gr-qc/9306005].

[7] J.C. Hackworth and E.J. Weinberg, *Oscillating bounce solutions and vacuum tunneling in de Sitter spacetime*, *Phys. Rev.* **D 71** (2005) 044014 [hep-th/0410142].

[8] S.R. Coleman, *The fate of the false vacuum, 1. Semiclassical theory*, *Phys. Rev.* **D 15** (1977) 2929.

[9] T. Banks, *More thoughts on the quantum theory of stable de Sitter space*, hep-th/0503066.

[10] T. Banks, *Cosmological breaking of supersymmetry or little lambda goes back to the future, II*, hep-th/0007146.

[11] L. Dyson, M. Kleban and L. Susskind, *Disturbing implications of a cosmological constant*, *JHEP* **10** (2002) 011 [hep-th/0208013].

[12] R. Penrose, *A complete guide to the laws of the universe*, Vintage, London, 2005.

[13] A. Albrecht and L. Sorbo, *Can the universe afford inflation?*, *Phys. Rev.* **D 70** (2004) 063528 [hep-th/0405270].

[14] B. Freivogel et al., *Inflation in AdS/CFT*, *JHEP* **03** (2006) 007 [hep-th/0510046].

[15] A. Aguirre and M.C. Johnson, *Two tunnels to inflation*, *Phys. Rev.* **D 73** (2006) 123529 [gr-qc/0512034].

[16] T. Banks and W. Fischler, *The holographic approach to cosmology*, hep-th/0412097; *Holographic cosmology*, hep-th/0405200; *Holographic cosmology 3.0*, *Phys. Scripta* **T117** (2005) 56–63 [hep-th/0310288]; *An holographic cosmology*, hep-th/0111142.

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